Magellan Mathematical Model

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Retyped by R. Kirk—5 November, 1998

The mathematical model required for MC&G (mapping, charting and geodesy) applications by USGS must be capable of determining the pixel coordinates (C_1,C_2) on the sinusoidal projection of a point on the venusian surface whose body fixed, VBF-85 coordinates (X,Y,Z) are given.

The *key assumption* upon which the formulation is based is that the (C_1,C_2) pixel coordinates of an imaged point may be associated with a single burst whose number is contained in the data annotation label (SDPS 101, Rev. 2, pp. 3–16), and that the equations and parameters associated with the burst (SDPS 101, Rev. 2, pp. F-56–57) may be used to compute the unambiguous range and doppler frequency at the burst reference time.

The projection of surface coordinates to pixel coordinates takes place in three primary steps:

- determine the burst number,
- compute the range and doppler frequency, and
- convert the range and doppler to sinusoidal coordinates.

Each of these steps is broken into smaller steps and presented in the following paragraphs. An error model is outlined in the last paragraph. The notation used attempts to follow that of SDPS 101.

All units are in the mks system and angles are expressed in radians unless otherwise noted.

A Determination of Burst Number

The burst number is required in order to access the correct set of SDPS processing parameters. These parameters include the spacecraft position and velocity vectors and the range and azimuth resampling coefficients required to convert range and doppler into the sinusoidal projection.

In regard to the triangulation mathematical model, the (C_1,C_2) measured coordinates are given. From these the burst number is determined. The computation is presented in the following paragraph.

A.1Determination of Burst Number from C_1-C_2

Each logical image data record in FILE 15 (multi-look image data in sinusoidal projection) contains the C_1-C_2 coordinates of the center of the first pixel in a line and the number of lines (*nl*) and the number of pixels per line (*np*) along with the burst counter in the data annotation label. Therefore, each record has a maximum and minimum C_1 and C_2 coordinate given by

$$C_1(\max) = \text{reference point offset in lines}*$$

$$C_2(\min) = \text{reference point offset in pixels}*$$

$$C_1(\min) = C_1(\max) - nl$$

$$C_2(\max) = C_2(\min) + np$$

The burst number (b) is found in the record which satisfies the condition

$$C_1(\min) \le C_1 \le C_1(\max)$$

$$C_2(\min) \le C_2 \le C_2(\max).$$

* from data annotation label in record.

B Computation of Range & Doppler Frequency

The apparent range is determined in two steps, which are:

B.1 computation of the geometric slant range, and

B.2 determination of the apparent range by applying the refraction correction

The apparent doppler frequency is computed in three steps:

B.3 geometric inertial range rate,

B.4 apparent range rate, and

B.5 computation of doppler frequency

B.1 Computation of the Geometric Slant Range

Having determined the burst number (b), the spacecraft position vector $\vec{\mathbf{x}}(t_b)$ and velocity vector $\vec{\mathbf{v}}(t_b)$ in VBF-85 coordinates are extracted from the SDPS processing parameters for burst b from FILE 16.

The vector coordinates are defined by

$$(X_S, Y_S, Z_S) \equiv \vec{\mathbf{x}}_S(t_b)$$

$$(\dot{X}_S, \dot{Y}_S, \dot{Z}_S) \equiv \vec{\mathbf{v}}_S(t_b)$$

And the geometric slant range is given by

$$R_g = \sqrt{(X_S - X)^2 + (Y_S - Y)^2 + (Z_S - Z)^2}$$

B.2 Computation of Apparent Range

The apparent range is determined by applying a refraction correction to the geometric slant range. The project specified atmosphere contains 86 layers extending from the planet surface – 10 km to an altitude of 76 km. The index of refraction is given for each layer. The equations given in SDPS 101 trace the ray path from the radar to the planet surface. A math model for USGS MC&G requires a ray trace when both the locations of the surface point and spacecraft are given.

The geometric slant range will be corrected resulting in the apparent range (R_a).

A rational polynomial function will be developed which gives the refraction correction as a function of the surface point elevation (h), the satellite altitude (H), and the geometric slant range R_g.

Figure B.2–1 depicts the venusian atmosphere reaching from the surface – 10 km to a height of 76 km. The atmosphere is divided into 87 shells (the first shell being free space). The index of refraction for shell k is represented by η_k.

Figure B.2–2 depicts the path of a radar pulse through the atmosphere. One is given the nadir distance of the satellite (Θ_1), the height of the satellite (H), and the height of the surface point (h).

The algorithm for finding the slant range correction is as follows:

1. determine the slant range to the top of the atmosphere using the law of cosines

$$R_1^0 = \tfrac{1}{2}(-B - \sqrt{B^2 - 4C}),\ \text{where}$$
$$B = \text{-}2(6051 + H)\cos\Theta_1$$
$$C = (6051 + H)^2 - (6127)^2$$

2. determine the angle of incidence at the top of the atmosphere and the angle subtended at Venus's center
$$\mathrm{I}_1 = \sin^{-1}(\tfrac{6051+H}{6127}\sin\Theta_1)$$
$$\gamma_1 = \sin^{-1}(\tfrac{R_1^2}{6127}\sin\Theta_1)$$
3. the path is now traced through the atmosphere to the shell having the elevation (h) where h is in integral km.

Set $R_a = R_1^0$ and $\gamma = \gamma_1$

FOR $k = 1, 2, \ldots, (76 - h)$
$$R_k = 6128 - k$$
$$R_{k+1} = R_k - 1$$
$$\Theta_{k+1} = \sin^{-1}(\eta_k \sin \mathrm{I}_k / \eta_{k+1})$$
$$\mathrm{I}_{k+1} = \sin^{-1}(R_k \sin\Theta_{k+1} / R_{k+1})$$
$$\gamma_{k+1} = \mathrm{I}_{k+1} - \Theta_{k+1}$$
$$d_{k+1} = R_{k+1}\sin\gamma_{k+1} / \sin\Theta_{k+1}$$
$$R_a = R_a + \eta_{k+1} d_{k+1}$$
$$\gamma = \gamma + \gamma_{k+1}$$
4. compute the slant range corresponding to Θ_1, h, and H.
$$R_s = \sqrt{(6051 + h)^2 + (6051 + H)^2 - 2(6051 + h)(6051 + H)\cos\gamma}$$
5. determine the slant range correction
$$\delta R = R_a - R_s$$

The algorithm outlined above is used to generate a large set of values $(R_s, H, h, \delta R)_j$, j=1, 2, …, n over the entire working range of Θ_1, h, and H. The values are then fit using a rational polynomial R
$$\delta R_j = \mathsf{R}(R_{s_j}, H_j, h_j)$$
The function R has the form
$$\delta R = \frac{a_0 + \mathbf{G}^T\mathbf{A}}{1 + \mathbf{G}^T\mathbf{B}}$$
in which
$$\mathbf{G}^T = \langle R_s, H, h, R_s H, R_s h, Hh, \ldots \rangle$$
$$\mathbf{A} = \langle a_1, a_2, \ldots, a_n \rangle^T$$
$$\mathbf{B} = \langle b_1, b_2, \ldots, b_n \rangle^T$$
The coefficients a_λ, b_λ are solved for.

Having once determined the function R and the coefficients a_λ, b_λ one can then correct the geometric slant range computed in paragraph B.1 for refraction:

$$\delta R_g = \mathrm{R}(R_g, H, h)$$

where

$$H = \sqrt{X_s^2 + Y_s^2 + Z_s^2} - 6051$$

$$h = \sqrt{X^2 + Y^2 + Z^2} - 6051.$$

B.3 Computation of Geometric Inertial Range Rate

The geometric inertial range rate is required for the computation of the doppler frequency. This range rate describes the relative motion of the satellite and surface point, both of which are moving in inertial space.

Consider (X_S,Y_S,Z_S) and (X,Y,Z) to be the instantaneous inertial coordinates of the spacecraft and surface point respectively, then the range rate is given by

$$\dot{R}_g = \tfrac{1}{R_g}[(X_S - X)(\dot{X}_{SI} - \dot{X}_I) + (Y_S - Y)(\dot{Y}_{SI} - \dot{Y}_I) + (Z_S - Z)(\dot{Z}_{SI} - \dot{Z}_I)].$$

The inertial velocities of the spacecraft and surface point are given by

$$\begin{Bmatrix} \dot{X}_{SI} \\ \dot{Y}_{SI} \end{Bmatrix} = \begin{Bmatrix} \dot{X}_S - \dot{\omega} Y_S \\ \dot{Y}_S + \dot{\omega} X_S \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{X}_I \\ \dot{Y}_I \end{Bmatrix} = \begin{Bmatrix} -\dot{\omega} Y \\ \dot{\omega} X \end{Bmatrix}$$

where $\dot{\omega}$ is the project specified rotational rate of the planet.

B.4 Computation of the Apparent Range Rate

The path that is changing and which affects the doppler frequency is the apparent path. The apparent range is given by

$$R_a = R_g + \delta R_g$$

where δR_g is the refraction correction which is a function of R_g. One may write

$$\dot{R}_a = \frac{dR_a}{dt} = \frac{dR_a}{dR_g}\frac{dR_g}{dt}$$

$$\dot{R}_a = \left(1 + \frac{d\delta R_g}{dR_g}\right)\dot{R}_g$$

Since δR_g is represented by a rational polynomial, the derivative is easily taken.

B.5 Computation of the Doppler Frequency

The frequency observed at the surface point is given by

$$f = f_0 + \tfrac{1}{\lambda}\dot{R}_a$$

Where f_0 and λ are the transmitted frequency and wavelength respectively. The frequency observed back at the satellite is

$$f_s = f + \tfrac{1}{\lambda}\dot{R}_a$$

Therefore

$$f_s = f_0 + \tfrac{2}{\lambda}\dot{R}_a$$

The doppler frequency is given by

$$f_d = f_0 - f_s$$
$$f_d = -\tfrac{2}{\lambda}\dot{R}_a$$

Note added 3/01: It is of course *critical* to use the exact radar wavelength in this calculation. We recently discovered that the code as originally written used the nominal wavelength of 12.5 cm. Substitution of the exact value c/2.3850 GHz = 12.569914 cm reduced discrepancies between computed doppler coordinates and those stored in the F-BIDRs, and eliminated problems with cursor jumping and artifacts in automatic stereomatching that initially affected the sensor model in the SOCET SET environment.

C. Conversion of Range and Doppler to Sinusoidal C_1-C_2 Coordinates

The equations of Appendix J of SDPS 101 will be used to convert (R_a,f_d) to (C_1,C_2) pixel coordinates. These equations utilize the SDPS processing parameters for burst b. In these equations $R=R_a$ and $f=f_d$.

D. Error Model

The preliminary error model will consist simply of adjusting the spacecraft ephemeris for an orbital pass. This is in addition to adjusting the surface point coordinates and minimizing the quadratic form of the C_1-C_2 residuals.
Corrections to the spacecraft ephemeris will be in terms of a single set of in-track, cross-track and radial errors $(\Delta I,\Delta C,\Delta R)$.
Corrections to the instantaneous inertial (VBF-85) position coordinates at every burst are given by

$$\begin{Bmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{Bmatrix} = \begin{bmatrix}\hat{\mathbf{u}}_I\\ \hat{\mathbf{u}}_C\\ \hat{\mathbf{u}}_R\end{bmatrix}^T \begin{Bmatrix}\Delta I\\ \Delta C\\ \Delta R\end{Bmatrix}$$

in which $\hat{\mathbf{u}}_I$, $\hat{\mathbf{u}}_C$, and $\hat{\mathbf{u}}_R$ are the unit vectors along the in-track, cross-track, and radial axes.
Vectors along the axes are given by

$$\vec{\mathbf{u}}_R = \langle X_S, Y_S, Z_S \rangle$$
$$\vec{\mathbf{v}}_{SI} = \langle \dot{X}_{SI}, \dot{Y}_{SI}, \dot{Z}_{SI} \rangle$$
$$\vec{\mathbf{u}}_C = \vec{\mathbf{u}}_R \times \vec{\mathbf{v}}_{SI}$$
$$\vec{\mathbf{u}}_I = \vec{\mathbf{u}}_C \times \vec{\mathbf{u}}_R.$$

The unit vectors are therefore given by

$$\hat{\mathbf{u}}_I = \vec{\mathbf{u}}_I / |\vec{\mathbf{u}}_I|$$
$$\hat{\mathbf{u}}_C = \vec{\mathbf{u}}_C / |\vec{\mathbf{u}}_C|$$
$$\hat{\mathbf{u}}_R = \vec{\mathbf{u}}_R / |\vec{\mathbf{u}}_R|.$$

Magellan Mathematical Model Partial Derivatives

Differential Derivation

Let:

$$u_1 = f - f_{d-g}(2,2)$$

$$u_2 = C_2 - C_{2g}(2,2)$$

$$u_3 = C_1 - C_{1g}(2,2)$$

$$\vec{\mathbf{U}}_1 = \left\langle 1, u_1, u_1^2 \right\rangle$$

$$\vec{\mathbf{U}}_2 = \left\langle 1, u_2, u_2^2 \right\rangle$$

$$\vec{\mathbf{U}}_3 = \left\langle 1, u_3, u_3^2 \right\rangle$$

The two condition equations of Appendix J are:

$$R - R_g(2,2) = \vec{\mathbf{U}}_1 \vec{\mathbf{U}}_g^T \vec{\mathbf{U}}_2^T,\ \vec{\mathbf{U}}_g \text{ constant}$$

$$f - f_{d-g}(2,2) = \vec{\mathbf{U}}_1 \vec{\mathbf{U}}_a^T \vec{\mathbf{U}}_2^T,\ \vec{\mathbf{U}}_a \text{ constant}$$

$$dR = \underbrace{d\vec{\mathbf{U}}_1}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g^T}_{3\times 3} \underbrace{\vec{\mathbf{U}}_2^T}_{3\times 1} + \underbrace{\vec{\mathbf{U}}_1}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g^T}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_2^T}_{3\times 1}$$

$$dR = \underbrace{\vec{\mathbf{U}}_2}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_1^T}_{3\times 1} + \underbrace{\vec{\mathbf{U}}_1}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g^T}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_2^T}_{3\times 1}$$

$$\underbrace{\vec{\mathbf{U}}_4}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_2}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g}_{3\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_5}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_1}_{1\times 3} \underbrace{\vec{\mathbf{U}}_g^T}_{3\times 3}$$

$$dR = \vec{\mathbf{U}}_4 \begin{Bmatrix} 0 \\ 1 \\ 2u_1 \end{Bmatrix} df + \vec{\mathbf{U}}_5 \begin{Bmatrix} 0 \\ 1 \\ 2u_2 \end{Bmatrix} dC_2$$

$$u_6 = \underbrace{\vec{\mathbf{U}}_4}_{1\times 3} \begin{Bmatrix} 0 \\ 1 \\ 2u_1 \end{Bmatrix}$$

$$u_7 = \underbrace{\vec{\mathbf{U}}_5}_{1\times 3} \begin{Bmatrix} 0 \\ 1 \\ 2u_2 \end{Bmatrix}$$

$$dR = u_6 df + u_7 dC_2$$

$$df = \underbrace{d\vec{\mathbf{U}}_3}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a^T}_{3\times 3} \underbrace{\vec{\mathbf{U}}_2^T}_{3\times 1} + \underbrace{\vec{\mathbf{U}}_3}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a^T}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_2^T}_{3\times 1}$$

$$df = \underbrace{\vec{\mathbf{U}}_2}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_3^T}_{3\times 1} + \underbrace{\vec{\mathbf{U}}_3}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a^T}_{3\times 3} \underbrace{d\vec{\mathbf{U}}_2^T}_{3\times 1}$$

$$\underbrace{\vec{\mathbf{U}}_8}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_2}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a}_{3\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_9}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_1}_{1\times 3} \underbrace{\vec{\mathbf{U}}_a^T}_{3\times 3}$$

$$df = \vec{\mathbf{U}}_8 \begin{Bmatrix} 0 \\ 1 \\ 2u_3 \end{Bmatrix} dC_1 + \vec{\mathbf{U}}_9 \begin{Bmatrix} 0 \\ 1 \\ 2u_2 \end{Bmatrix} dC_2$$

$$u_{10} = \underbrace{\vec{\mathbf{U}}_8}_{1\times3} \begin{Bmatrix} 0 \\ 1 \\ 2u_3 \end{Bmatrix}$$

$$u_{11} = \underbrace{\vec{\mathbf{U}}_9}_{1\times3} \begin{Bmatrix} 0 \\ 1 \\ 2u_2 \end{Bmatrix}$$

$$df = u_{10} dC_1 + u_{11} dC_2$$

Now substituting df

$$dR = u_6 df + u_7 dC_2$$

$$dR = u_6 (u_{10} dC_1 + u_{11} dC_2) + u_7 dC_2$$

$$dR = u_6 u_{10} dC_1 + (u_6 u_{11} + u_7) dC_2$$

$$u_{12} = u_6 u_{10}$$

$$u_{13} = u_6 u_{11} + u_7$$

$$dR = u_{12} dC_1 + u_{13} dC_2$$

now

$$\begin{Bmatrix} dR \\ df \end{Bmatrix} = \begin{bmatrix} u_{12} & u_{13} \\ u_{10} & u_{11} \end{bmatrix} \begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \frac{1}{u_{12} u_{11} - u_{10} u_{13}} \begin{bmatrix} u_{11} & -u_{13} \\ -u_{10} & u_{12} \end{bmatrix} \begin{Bmatrix} dR \\ df \end{Bmatrix}$$

$$\underbrace{\vec{U}_{14}}_{2\times2} = \frac{1}{u_{12} u_{11} - u_{10} u_{13}} \begin{bmatrix} u_{11} & -u_{13} \\ -u_{10} & u_{12} \end{bmatrix}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{U}_{14} \begin{Bmatrix} dR \\ df \end{Bmatrix}$$

$$dR = dR_g + d\delta R$$

$$df = -\tfrac{2}{\lambda} d\dot{R}$$

$$d\dot{R} = \left(1 + \frac{d\delta R}{dR_g}\right) d\dot{R}_g + \dot{R}_g d\left(\frac{d\delta R}{dR_g}\right)$$

$$u_{15} = -\tfrac{2}{\lambda}\left(1 + \frac{d\delta R}{dR_g}\right)$$

$$u_{16} = -\tfrac{2}{\lambda} \dot{R}_g$$

$$df = u_{15} d\dot{R}_g + u_{16} d\left(\frac{d\delta R}{dR_g}\right)$$

$$\begin{Bmatrix} dR \\ df \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u_{15} \end{bmatrix} \begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & u_{16} \end{bmatrix} \begin{Bmatrix} d\delta R \\ d(d\delta R / dR_g) \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{17}}_{2\times 2} = \underbrace{\vec{\mathbf{U}}_{14}}_{2\times 2} \begin{bmatrix} 1 & 0 \\ 0 & u_{15} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{18}}_{2\times 2} = \underbrace{\vec{\mathbf{U}}_{14}}_{2\times 2} \begin{bmatrix} 1 & 0 \\ 0 & u_{16} \end{bmatrix}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{\mathbf{U}}_{17} \begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix} + \vec{\mathbf{U}}_{18} \begin{Bmatrix} d\delta R \\ d(d\delta R / dR_g) \end{Bmatrix}$$

Let

$$\underbrace{\mathbf{G}_1}_{2\times 3} = \frac{\partial(\delta R_1\, d\delta R / dR_g)}{d(X, Y, Z)}$$

$$\underbrace{\mathbf{G}_2}_{2\times 3} = \frac{\partial(\delta R_1\, d\delta R / dR_g)}{d(X_s^0, Y_s^0, Z_s^0)}$$

output from the refraction function.

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{\mathbf{U}}_{17} \begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix} + \vec{\mathbf{U}}_{18} \mathbf{G}_1 \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{18} \mathbf{G}_2 \begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{19}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{18}}_{2\times 2} \underbrace{\mathbf{G}_1}_{2\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{20}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{18}}_{2\times 2} \underbrace{\mathbf{G}_2}_{2\times 3}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{\mathbf{U}}_{17} \begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix} + \vec{\mathbf{U}}_{19} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{20} \begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix}$$

$$\vec{\mathbf{U}}_{21} = \left\langle (X_S^0 - X), (Y_S^0 - Y), (Z_S^0 - Z) \right\rangle$$

$$\vec{\mathbf{U}}_{22} = \left\langle (\dot{X}_{SI} - \dot{X}_I), (\dot{Y}_{SI} - \dot{Y}_I), \dot{Z}_{SI} \right\rangle$$

$$\dot{R}_g = \tfrac{1}{R_g} \vec{\mathbf{U}}_{21} \vec{\mathbf{U}}_{22}^T$$

$$d\dot{R}_g = -\tfrac{dR_g}{R_g^2} \vec{\mathbf{U}}_{21} \vec{\mathbf{U}}_{22}^T + \underbrace{\tfrac{1}{R_g} d\vec{\mathbf{U}}_{21} \vec{\mathbf{U}}_{22}^T}_{\frac{1}{R_g} \vec{\mathbf{U}}_{22} d\vec{\mathbf{U}}_{21}^T} + \tfrac{1}{R_g} \vec{\mathbf{U}}_{21} d\vec{\mathbf{U}}_{22}^T$$

$$u_{23} = -\tfrac{1}{R_g^2} \vec{\mathbf{U}}_{21} \vec{\mathbf{U}}_{22}^T$$

$$\underbrace{\vec{\mathbf{U}}_{24}}_{1\times 3} = \tfrac{1}{R_g}\underbrace{\vec{\mathbf{U}}_{22}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{25}}_{1\times 3} = \tfrac{1}{R_g}\underbrace{\vec{\mathbf{U}}_{21}}_{1\times 3}$$

$$d\dot{R}_g = u_{23} dR_g + \vec{\mathbf{U}}_{24}\begin{Bmatrix} dX_S^0 - dX \\ dY_S^0 - dY \\ dZ_S^0 - dZ \end{Bmatrix} + \vec{\mathbf{U}}_{25}\begin{Bmatrix} d\dot{X}_{SI} - d\dot{X}_I \\ d\dot{Y}_{SI} - d\dot{Y}_I \\ d\dot{Z}_{SI} \end{Bmatrix}$$

$$d\dot{X}_{SI} = d\dot{Y}_{SI} = d\dot{Z}_{SI} = 0$$

$$d\dot{X}_I = -\dot{\omega}\, dY$$

$$d\dot{Y}_I = \dot{\omega}\, dX$$

$$\begin{Bmatrix} -d\dot{X}_I \\ -d\dot{Y}_I \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & +\dot{\omega} & 0 \\ -\dot{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{26}}_{1\times 3} = -\underbrace{\vec{\mathbf{U}}_{24}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{27}}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_{25}}_{1\times 3}\underbrace{\begin{bmatrix} 0 & +\dot{\omega} & 0 \\ -\dot{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{3\times 3}$$

$$d\dot{R}_g = u_{23} dR_g + \vec{\mathbf{U}}_{24}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{26}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{27}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{28}}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_{26}}_{1\times 3} + \underbrace{\vec{\mathbf{U}}_{27}}_{1\times 3}$$

$$d\dot{R}_g = u_{23} dR_g + \vec{\mathbf{U}}_{24}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{28}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$R_g^2 = \vec{\mathbf{U}}_{21}\vec{\mathbf{U}}_{21}^T$$

$$2R_g dR_g = d\vec{\mathbf{U}}_{21}\vec{\mathbf{U}}_{21}^T + \vec{\mathbf{U}}_{21} d\vec{\mathbf{U}}_{21}^T$$

$$2R_g dR_g = 2\vec{\mathbf{U}}_{21} d\vec{\mathbf{U}}_{21}^T$$

$$dR_g = \tfrac{1}{R_g}\vec{\mathbf{U}}_{21}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} - \tfrac{1}{R_g}\vec{\mathbf{U}}_{21}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{29}}_{1\times 3} = \tfrac{1}{R_g}\vec{\mathbf{U}}_{21}$$

$$\vec{\mathbf{U}}_{30} = -\vec{\mathbf{U}}_{29}$$

$$dR_g = \vec{\mathbf{U}}_{29}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{30}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$d\dot{R}_g = u_{23}\vec{\mathbf{U}}_{29}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + u_{23}\vec{\mathbf{U}}_{30}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{24}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{28}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{31}}_{1\times 3} = u_{23}\vec{\mathbf{U}}_{29} + \vec{\mathbf{U}}_{24}$$

$$\underbrace{\vec{\mathbf{U}}_{32}}_{1\times 3} = u_{23}\vec{\mathbf{U}}_{30} + \vec{\mathbf{U}}_{28}$$

$$d\dot{R}_g = \vec{\mathbf{U}}_{31}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{32}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

Then

$$\underbrace{\begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} \vec{\mathbf{U}}_{29} \\ \vec{\mathbf{U}}_{31} \end{bmatrix}}_{2\times 3}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \underbrace{\begin{bmatrix} \vec{\mathbf{U}}_{30} \\ \vec{\mathbf{U}}_{32} \end{bmatrix}}_{2\times 3}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{33}}_{2\times 3} = \begin{bmatrix} \vec{\mathbf{U}}_{29} \\ \vec{\mathbf{U}}_{31} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{34}}_{2\times 3} = \begin{bmatrix} \vec{\mathbf{U}}_{30} \\ \vec{\mathbf{U}}_{32} \end{bmatrix}$$

$$\begin{Bmatrix} dR_g \\ d\dot{R}_g \end{Bmatrix} = \vec{\mathbf{U}}_{33}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{34}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{\mathbf{U}}_{17}\vec{\mathbf{U}}_{33}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{17}\vec{\mathbf{U}}_{34}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{19}\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} + \vec{\mathbf{U}}_{20}\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{35}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{17}}_{2\times 2}\underbrace{\vec{\mathbf{U}}_{33}}_{2\times 3} + \underbrace{\vec{\mathbf{U}}_{20}}_{2\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{36}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{17}}_{2\times 2}\underbrace{\vec{\mathbf{U}}_{34}}_{2\times 3} + \underbrace{\vec{\mathbf{U}}_{19}}_{2\times 3}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \vec{\mathbf{U}}_{35} \begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} + \vec{\mathbf{U}}_{36} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\begin{Bmatrix} dX_S^0 \\ dY_S^0 \\ dZ_S^0 \end{Bmatrix} = \begin{bmatrix} \hat{\mathbf{u}}_I \\ \hat{\mathbf{u}}_C \\ \hat{\mathbf{u}}_R \end{bmatrix}^T \begin{Bmatrix} d\Delta I_S \\ d\Delta C_S \\ d\Delta R_S \end{Bmatrix}$$

$$\underbrace{\dot{\mathbf{B}}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{35}}_{2\times 3} \underbrace{\begin{bmatrix} \hat{\mathbf{u}}_I \\ \hat{\mathbf{u}}_C \\ \hat{\mathbf{u}}_R \end{bmatrix}^T}_{3\times 3}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \dot{\mathbf{B}} \begin{Bmatrix} d\Delta I_S \\ d\Delta C_S \\ d\Delta R_S \end{Bmatrix} + \vec{\mathbf{U}}_{36} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = (R_v + h) \begin{Bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{Bmatrix}$$

$$\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \begin{Bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{Bmatrix} dh + (R_v + h) \begin{Bmatrix} -\sin\varphi\cos\lambda d\varphi - \cos\varphi\sin\lambda d\lambda \\ -\sin\varphi\sin\lambda d\varphi + \cos\varphi\cos\lambda d\lambda \\ \cos\varphi d\varphi \end{Bmatrix}$$

$$\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \begin{bmatrix} -(R_v + h)\sin\varphi\cos\lambda & -(R_v + h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(R_v + h)\sin\varphi\sin\lambda & (R_v + h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (R_v + h) & 0 & \sin\varphi \end{bmatrix} \begin{Bmatrix} d\varphi \\ d\lambda \\ dh \end{Bmatrix}$$

$$\vec{\mathbf{U}}_{37} = \begin{bmatrix} -(R_v + h)\sin\varphi\cos\lambda & -(R_v + h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(R_v + h)\sin\varphi\sin\lambda & (R_v + h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (R_v + h) & 0 & \sin\varphi \end{bmatrix}$$

$$\underbrace{\ddot{\mathbf{B}}}_{2\times 3} = \vec{\mathbf{U}}_{36} \vec{\mathbf{U}}_{37}$$

$$\begin{Bmatrix} dC_1 \\ dC_2 \end{Bmatrix} = \dot{\mathbf{B}} \begin{Bmatrix} d\Delta I_S \\ d\Delta C_S \\ d\Delta R_S \end{Bmatrix} + \ddot{\mathbf{B}} \begin{Bmatrix} d\varphi \\ d\lambda \\ dh \end{Bmatrix}$$

Magellan Mathematical Model Algorithm

Input

(φ, λ, h) Latitude, longitude, and height of surface point.
$(X_S, Y_S, Z_S, \dot{X}_S, \dot{Y}_S, \dot{Z}_S)$ VBF-85 spacecraft position/velocity.
R_V mean radius of Venus.
$\dot{\omega}$ rotational rate of Venus.
$(\Delta I_S, \Delta C_S, \Delta R_S)$ corrections to spacecraft position.

$$\left.\begin{array}{l} \mu_{g-ij}, \mu_{a-ij}, f_{d-g}(2,2), R_g(2,2) \\ C_{2g}(2,2), C_{1g}(2,2) \end{array}\right\} \text{processing parameters}$$

λ radar wavelength.
S_2 = -1 for sinusoidal projection, +1 for oblique sinusoidal projection.

Output

(C_1, C_2) image coordinates

$$\left.\begin{array}{l} \underbrace{\dot{\mathbf{B}}}_{2\times 3} = \dfrac{\partial(C_1, C_2)}{\partial(\Delta I_S, \Delta C_S, \Delta R_S)} \\ \underbrace{\ddot{\mathbf{B}}}_{2\times 3} = \dfrac{\partial(C_1, C_2)}{\partial(\varphi, \lambda, h)} \end{array}\right\} \text{partial derivatives}$$

All units are meters, kg, seconds, radians.

Projective Equations Algorithm

1. VBF-85 coordinates of the surface point

$$X = (R_V + h)\cos\varphi\cos\lambda$$
$$Y = (R_V + h)\cos\varphi\sin\lambda$$
$$Z = (R_V + h)\sin\varphi$$

2. Unit vectors in the spacecraft in-track, cross-track, and radial system

$$\dot{X}_{SI} = \dot{X}_S - \dot{\omega} Y_S$$
$$\dot{Y}_{SI} = \dot{Y}_S + \dot{\omega} X_S$$
$$\dot{Z}_{SI} = \dot{Z}_S$$
$$\vec{\mathbf{U}}_R = \left\langle X_S, Y_S, Z_S \right\rangle$$
$$\vec{\mathbf{V}}_{SI} = \left\langle \dot{X}_{SI}, \dot{Y}_{SI}, \dot{Z}_{SI} \right\rangle$$
$$\vec{\mathbf{U}}_C = \vec{\mathbf{U}}_R \times \vec{\mathbf{V}}_{SI}$$
$$\vec{\mathbf{U}}_I = \vec{\mathbf{U}}_C \times \vec{\mathbf{U}}_R$$

$$\hat{\mathbf{u}}_I = \vec{\mathbf{U}}_I / \left|\vec{\mathbf{U}}_I\right|$$

$$\hat{\mathbf{u}}_C = \vec{\mathbf{U}}_C / \left|\vec{\mathbf{U}}_C\right|$$

$$\hat{\mathbf{u}}_R = \vec{\mathbf{U}}_R / \left|\vec{\mathbf{U}}_R\right|$$

3. Corrected VBF-85 spacecraft coordinates

$$\underbrace{\begin{Bmatrix} X_S^0 \\ Y_S^0 \\ Z_S^0 \end{Bmatrix}}_{3\times1} = \underbrace{\begin{Bmatrix} X_S \\ Y_S \\ Z_S \end{Bmatrix}}_{3\times1} + \underbrace{\begin{bmatrix} \hat{\mathbf{u}}_I \\ \hat{\mathbf{u}}_C \\ \hat{\mathbf{u}}_R \end{bmatrix}^T}_{3\times3} \underbrace{\begin{Bmatrix} \Delta I_S \\ \Delta C_S \\ \Delta R_S \end{Bmatrix}}_{3\times1}$$

4. Geometric slant range

$$R_g = \sqrt{(X_S^0 - X)^2 + (Y_S^0 - Y)^2 + (Z_S^0 - Z)^2}$$

5. Refraction correction (functions provided in appendix)

Let

$$\vec{\mathbf{U}}_S^0 = \left\langle X_S^0, Y_S^0, Z_S^0 \right\rangle$$

$$\vec{\mathbf{P}} = \langle X, Y, Z \rangle$$

Then

$$\delta R = \mathsf{R}(R_V, \vec{\mathbf{U}}_S^0, \vec{\mathbf{P}})$$

$$\frac{d\delta R}{dR_g} = \mathsf{R}'(R_V, \vec{\mathbf{U}}_S^0, \vec{\mathbf{P}})$$

6. Apparent slant range

$$R = R_g + \delta R$$

7.Geometric inertial range rate

$$\dot{X}_I = -\dot{\omega} Y$$

$$\dot{Y}_I = \dot{\omega} X$$

$$\dot{R}_g = \tfrac{1}{R_g} \underbrace{\left\langle (X_S^0 - X), (Y_S^0 - Y), (Z_S^0 - Z) \right\rangle}_{1\times3} \underbrace{\begin{Bmatrix} \dot{X}_{SI} - \dot{X}_I \\ \dot{Y}_{SI} - \dot{Y}_I \\ \dot{Z}_{SI} - \dot{Z}_I \end{Bmatrix}}_{3\times1}$$

8.Apparent range rate

$$\dot{R} = \left(1 + \tfrac{d\delta R}{dR_g}\right) \dot{R}_g$$

9.Doppler frequency

$$f = -\tfrac{2}{\lambda}\dot{R}$$

10. C_1 and C_2 coordinates

Define:

$$\underbrace{\vec{\mathbf{U}}_g}_{3\times3} = \left[\mu_{g-ij}\right]$$

$$\underbrace{\vec{\mathbf{U}}_a}_{3\times3} = \left[\mu_{a-ij}\right]$$

$$\tilde{u}_1 = f - f_{d-g}(2,2)$$

$$\underbrace{\tilde{\mathbf{U}}_1}_{3\times1} = \left\langle 1, \tilde{u}_1 . \tilde{u}_1^2 \right\rangle^T$$

$$\Delta R = R - R_g(2,2)$$

$$\begin{Bmatrix} C_1 \\ B_1 \\ A_1 \end{Bmatrix} = \underbrace{\vec{\mathbf{U}}_g}_{3\times3}\underbrace{\tilde{\mathbf{U}}_1}_{3\times1} - \begin{Bmatrix} \Delta R \\ 0 \\ 0 \end{Bmatrix}$$

$$C_2 = \frac{-B_1 - S_1\sqrt{B_1^2 - 4A_1C_1}}{2A_1} - C_{2g}(2,2), \quad S_1 = 1$$

$$\tilde{u}_2 = C_2 - C_{2g}(2,2)$$

$$\underbrace{\vec{\mathbf{U}}_2}_{3\times1} = \left\langle 1, \tilde{u}_2 . \tilde{u}_2^2 \right\rangle^T$$

$$\Delta f = f - f_{d-g}(2,2)$$

$$\begin{Bmatrix} C_2 \\ B_2 \\ A_2 \end{Bmatrix} = \underbrace{\vec{\mathbf{U}}_a}_{3\times3}\underbrace{\tilde{\mathbf{U}}_2}_{3\times1} - \begin{Bmatrix} \Delta f \\ 0 \\ 0 \end{Bmatrix}$$

$$C_1 = \frac{-B_2 - S_2\sqrt{B_2^2 - 4A_2C_2}}{2A_2} - C_{1g}(2,2), \quad S_1 = 1$$

Partial Derivatives Algorithm

$$u_1 = f - f_{d-g}(2,2)$$

$$u_2 = C_2 - C_{2g}(2,2)$$

$$u_3 = C_1 - C_{1g}(2,2)$$

$$\vec{\mathbf{U}}_1 = \left\langle 1, u_1, u_1^2 \right\rangle$$

$$\vec{\mathbf{U}}_2 = \left\langle 1, u_2, u_2^2 \right\rangle$$

$$\vec{\mathbf{U}}_3 = \left\langle 1, u_3, u_3^2 \right\rangle$$

$$\underbrace{\vec{\mathbf{U}}_4}_{1\times3} = \underbrace{\vec{\mathbf{U}}_2}_{1\times3}\underbrace{\vec{\mathbf{U}}_g}_{3\times3}$$

$$\underbrace{\vec{\mathbf{U}}_5}_{1\times3} = \underbrace{\vec{\mathbf{U}}_1}_{1\times3}\underbrace{\vec{\mathbf{U}}_g^T}_{3\times3}$$

$$u_6 = \underbrace{\vec{\mathbf{U}}_4}_{1\times3}\langle 0,1,2u_1\rangle^T$$

$$u_7 = \underbrace{\vec{\mathbf{U}}_5}_{1\times3}\langle 0,1,2u_2\rangle^T$$

$$\underbrace{\vec{\mathbf{U}}_8}_{1\times3} = \underbrace{\vec{\mathbf{U}}_2}_{1\times3}\underbrace{\vec{\mathbf{U}}_a}_{3\times3}$$

$$\underbrace{\vec{\mathbf{U}}_9}_{1\times3} = \underbrace{\vec{\mathbf{U}}_3}_{1\times3}\underbrace{\vec{\mathbf{U}}_a^T}_{3\times3}$$

$$u_{10} = \underbrace{\vec{\mathbf{U}}_8}_{1\times3}\langle 0,1,2u_3\rangle^T$$

$$u_{11} = \underbrace{\vec{\mathbf{U}}_9}_{1\times3}\langle 0,1,2u_2\rangle^T$$

$$u_{12} = u_6 u_{10}$$

$$u_{13} = u_6 u_{11} + u_7$$

$$\underbrace{\vec{\mathbf{U}}_{14}}_{2\times2} = \frac{1}{u_{11}u_{12} - u_{10}u_{13}}\begin{bmatrix} u_{11} & u_{13} \\ -u_{10} & u_{12} \end{bmatrix}$$

$$u_{15} = -\tfrac{2}{\lambda}\left(1 + \tfrac{d\delta R}{dR_g}\right)$$

$$u_{16} = -\tfrac{2}{\lambda}\dot{R}$$

$$\underbrace{\vec{\mathbf{U}}_{17}}_{2\times2} = \underbrace{\vec{\mathbf{U}}_{14}}_{2\times2}\begin{bmatrix} 1 & 0 \\ 0 & u_{15} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{18}}_{2\times2} = \underbrace{\vec{\mathbf{U}}_{14}}_{2\times2}\begin{bmatrix} 1 & 0 \\ 0 & u_{16} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{19}}_{2\times3} = \underbrace{\vec{\mathbf{U}}_{18}}_{2\times2}\underbrace{\mathbf{G}_1}_{2\times3}$$

$$\underbrace{\vec{\mathbf{U}}_{20}}_{2\times3} = \underbrace{\vec{\mathbf{U}}_{18}}_{2\times2}\underbrace{\mathbf{G}_2}_{2\times3}$$

$$\underbrace{\vec{\mathbf{U}}_{21}}_{1\times3} = \left\langle (X_S^0 - X),(Y_S^0 - Y),(Z_S^0 - Z)\right\rangle$$

$$\underbrace{\vec{\mathbf{U}}_{22}}_{1\times3} = \left\langle (\dot{X}_{SI} - \dot{X}_S),(\dot{Y}_{SI} - \dot{Y}_S),\dot{Z}_{SI}\right\rangle$$

$$u_{23} = -\tfrac{1}{R_g}\vec{\mathbf{U}}_{21}\vec{\mathbf{U}}_{22}^T$$

$$\underbrace{\vec{\mathbf{U}}_{24}}_{1\times 3} = \tfrac{1}{R_g}\underbrace{\vec{\mathbf{U}}_{22}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{25}}_{1\times 3} = \tfrac{1}{R_g}\underbrace{\vec{\mathbf{U}}_{21}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{26}}_{1\times 3} = -\underbrace{\vec{\mathbf{U}}_{24}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{27}}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_{25}}_{1\times 3}\begin{bmatrix} 0 & \dot{\omega} & 0 \\ -\dot{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{28}}_{1\times 3} = \underbrace{\vec{\mathbf{U}}_{26}}_{1\times 3} + \underbrace{\vec{\mathbf{U}}_{27}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{29}}_{1\times 3} = \tfrac{1}{R_g}\underbrace{\vec{\mathbf{U}}_{21}}_{1\times 3} \text{ (this certainly looks the same as } \vec{\mathbf{U}}_{25}\text{)}$$

$$\underbrace{\vec{\mathbf{U}}_{30}}_{1\times 3} = -\underbrace{\vec{\mathbf{U}}_{29}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{31}}_{1\times 3} = u_{22}\underbrace{\vec{\mathbf{U}}_{29}}_{1\times 3} + \underbrace{\vec{\mathbf{U}}_{24}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{32}}_{1\times 3} = u_{22}\underbrace{\vec{\mathbf{U}}_{30}}_{1\times 3} + \underbrace{\vec{\mathbf{U}}_{28}}_{1\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{33}}_{2\times 3} = \begin{bmatrix} \vec{\mathbf{U}}_{29} \\ \vec{\mathbf{U}}_{31} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{34}}_{2\times 3} = \begin{bmatrix} \vec{\mathbf{U}}_{30} \\ \vec{\mathbf{U}}_{32} \end{bmatrix}$$

$$\underbrace{\vec{\mathbf{U}}_{35}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{17}}_{2\times 2}\underbrace{\vec{\mathbf{U}}_{33}}_{2\times 3} + \underbrace{\vec{\mathbf{U}}_{20}}_{2\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{36}}_{2\times 3} = \underbrace{\vec{\mathbf{U}}_{17}}_{2\times 2}\underbrace{\vec{\mathbf{U}}_{34}}_{2\times 3} + \underbrace{\vec{\mathbf{U}}_{19}}_{2\times 3}$$

$$\dot{\mathbf{B}} = \underbrace{\vec{\mathbf{U}}_{35}}_{2\times 3}\underbrace{\begin{bmatrix} \hat{\mathbf{u}}_I \\ \hat{\mathbf{u}}_C \\ \hat{\mathbf{u}}_R \end{bmatrix}^T}_{3\times 3}$$

$$\underbrace{\vec{\mathbf{U}}_{37}}_{3\times 3} = \begin{bmatrix} -(R_V+h)\sin\varphi\cos\lambda & -(R_V+h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(R_V+h)\sin\varphi\sin\lambda & (R_V+h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (R_V+h)\cos\varphi & 0 & \sin\varphi \end{bmatrix}$$

$$\underbrace{\ddot{\mathbf{B}}}_{3\times 3} = \underbrace{\vec{\mathbf{U}}_{36}}_{3\times 3}\underbrace{\vec{\mathbf{U}}_{37}}_{3\times 3}$$

Appendix: Refraction

The refraction is represented by a rational polynomial expression whose coefficients are determined by fitting the refraction corrections for various geometric configurations representing the collection geometry.

The input its:

(X_S^0, Y_S^0, Z_S^0) the corrected spacecraft coordinates

(X, Y, Z) the surface point coordinates

R_V the mean radius of Venus

The output is:

δR the refraction correction to slant range

$d\delta R / dR_g$ the rate of change of δR with respect to slant range R_g

$$\left.\begin{aligned} \underbrace{\mathbf{G_1}}_{2\times 3} &= \frac{\partial(\delta R, d\delta R / dR_g)}{\partial(X, Y, Z)} \\ \underbrace{\mathbf{G}_2}_{2\times 3} &= \frac{\partial(\delta R, d\delta R / dR_g)}{\partial(X_S^0, Y_S^0, Z_S^0)} \end{aligned}\right\} \text{partial derivatives}$$